Q1

Express the following complex numbers in exponential form:

(i) $-5(\cos 2 - \mathrm{i}\sin 2)$

(ii) $(\sqrt{2}-\sqrt{6})-(\sqrt{2}+\sqrt{6})\mathrm{i}$

[4]

$\cos(\pi - x) = -\cos x$

$\sin(\pi - x) = \sin x$

$z = re^{i\theta}$, $\theta = \arg(z)$, $r = |z|$

$z = r(\cos\theta + i\sin\theta)$, $r = |z|$, $\theta = \arg(z)$

(i) $-5(\cos 2 - i\sin 2) = 5(-\cos 2 + i\sin 2)$

$= 5(\cos(\pi-2) + i\sin(\pi-2))$

$\boxed{= 5e^{(\pi-2)i}}$

(ii) $|\sqrt{2}-\sqrt{6}+(\sqrt{2}+\sqrt{6})i| = \sqrt{(\sqrt{2}-\sqrt{6})^2+(\sqrt{2}+\sqrt{6})^2}$

$= \sqrt{8-\cancel{2\sqrt{12}}+8+\cancel{2\sqrt{12}}} = \sqrt{16} = 4$

$\sqrt{2}-\sqrt{6}<0$

$(\sqrt{2}-\sqrt{6})-(\sqrt{2}+\sqrt{6})i = -(\sqrt{6}-\sqrt{2})-(\sqrt{6}+\sqrt{2})i$

$\Rightarrow$ number is in 3^{rd} quadrant

$\arg\left((\sqrt{2}-\sqrt{6})-(\sqrt{2}+\sqrt{6})i\right) = -\pi + \tan^{-1}\left(\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}\right)$

$= -\pi + \frac{5\pi}{12} = -\frac{7\pi}{12}$

$\boxed{4e^{-\frac{7\pi}{12}i}}$

Q2

$z_1 = 14\mathrm{e}^{9\mathrm{i}}$

$z_2 = 10\mathrm{e}^{-2\mathrm{i}}$

(i) Work out z_1z_2 and $\frac{z_2}{z_1}$, giving your answers in exponential form.

(ii) Express your answers to part (i) as complex numbers in modulus-argument form. In each case the modulus and argument should be given as exact values, with the argument θ being given in the interval $-\pi < \theta \le \pi$.

[4]

$z = re^{i\theta}$, $\theta = \arg(z)$, $r = |z|$

$z = r(\cos\theta + i\sin\theta)$, $r = |z|$, $\theta = \arg(z)$

Changing the argument of a complex number by multiples of 2π doesn't change the complex number

(i) $z_1z_2 = (14e^{9i})(10e^{-2i}) = 140e^{9i+(-2i)}$

$\boxed{z_1z_2 = 140e^{7i}}$

$\frac{z_2}{z_1} = \frac{10e^{-2i}}{14e^{9i}} = \frac{10}{14}e^{-2i-(9i)}$

$\boxed{\frac{z_2}{z_1} = \frac{5}{7}e^{-11i}}$

(ii) $\pi < 7 \Rightarrow 7$ is outside the interval

But $0 < 7-2\pi < \pi$

So $\boxed{z_1z_2 = 140(\cos(7-2\pi)+i\sin(7-2\pi))}$

$-11 < -\pi \Rightarrow -11$ is outside the interval

But $0 < -11+4\pi < \pi$

So $\boxed{\frac{z_2}{z_1} = \frac{5}{7}(\cos(4\pi-11)+i\sin(4\pi-11))}$

Q3a

Notes: ① The diagram here has z in the first quadrant, but similar triangles will result wherever you put z. Try it!

② One way to confirm that the triangles are similar is to use the cosine rule to find the length of the third side in each triangle. In the 0, 1, z triangle it is

$$\sqrt{1^2 + r^2 - 2r\cos\theta}$$

In the 0, z, z^2 triangle it is

$$\sqrt{r^2 + r^4 - 2r^3\cos\theta} = r\sqrt{1^2 + r^2 - 2r\cos\theta}$$

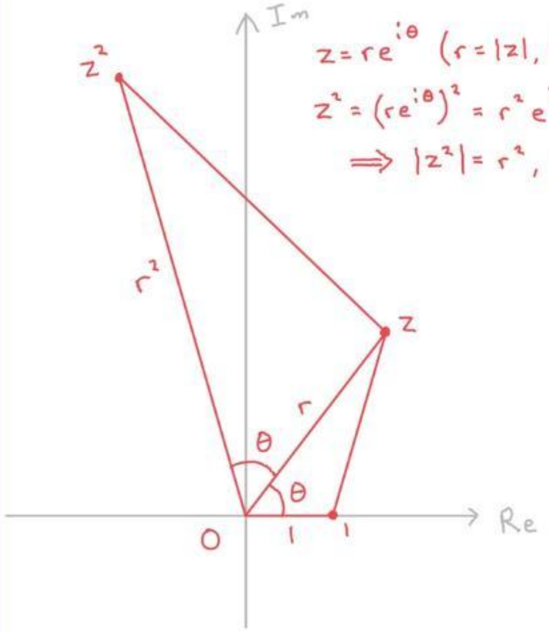
which again shows r as the scale factor.

a)

$z = re^{i\theta}$ $(r = |z|,\ \theta = \arg(z))$

$z^2 = (re^{i\theta})^2 = r^2 e^{i(2\theta)}$

$\Rightarrow |z^2| = r^2,\ \arg(z^2) = 2\theta$



The triangles formed by 0, 1, z and 0, z, z^2 are similar triangles with scale factor r. The side from 0 to 1 in the first triangle corresponds to the side from 0 to z in the second triangle.

Q3b

Given the points 1 and z on an Argand diagram, where $z \neq 0$ is a complex number, explain how to find each of the following points by geometrical construction. In each case provide a sketch to illustrate your answer.

(a) z^2

[3]

(b) $(2 - \mathrm{i})z$

[3]

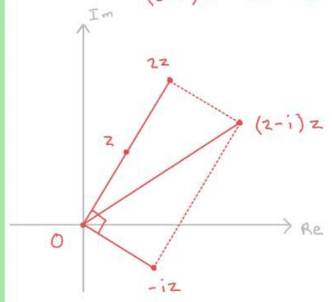
$-i = e^{-\frac{\pi}{2}i}$ and $z = re^{i\theta}$

So $-iz = re^{(\theta - \frac{\pi}{2})i}$

The change in argument by $-\frac{\pi}{2}$ indicates a clockwise rotation by $\frac{\pi}{2}$ radians (90°)

Note: As long as 0, z_1 and z_2 don't lie on a straight line, the four points 0, z_1, z_2 and $z_1 + z_2$ will form a parallelogram in an Argand diagram. In this particular case, of course, the parallelogram is a rectangle!

b) $(2-i)z = 2z - iz = 2z + (-iz)$



Enlarge by scale factor 2 (centred at origin) to map z to $2z$. Rotate $\pi/2$ radians clockwise about the origin to map z to $-iz$. The lines from O to $2z$ and from O to $-iz$ form two sides of a parallelogram. The fourth vertex of the parallelogram is $(2-i)z$.

Q4a

a)

Increasing the argument of a complex number by 2π radians means a rotation by 2π radians (360°) anticlockwise about the origin in an Argand diagram. This brings the number back to where it started, and so doesn't change the number.

It follows that $re^{i\theta}$ and $re^{i(\theta + 2\pi)}$ are equal.

Q4b

b) Let $\sqrt{z} = pe^{i\alpha}$

$(\sqrt{z})^2 = z$, therefore

$(pe^{i\alpha})^2 = p^2 e^{i(2\alpha)} = re^{i\theta}$

So $p^2 = r \Rightarrow p = \sqrt{r}$, $2\alpha = \theta \Rightarrow \alpha = \frac{\theta}{2}$

OR $(pe^{i\alpha})^2 = p^2 e^{i(2\alpha)} = re^{i(\theta + 2\pi)}$

So $p^2 = r \Rightarrow p = \sqrt{r}$, $2\alpha = \theta + 2\pi \Rightarrow \alpha = \frac{\theta}{2} + \pi$

$$\boxed{\sqrt{z} = \sqrt{r}\, e^{i\frac{\theta}{2}} \quad \text{or} \quad \sqrt{r}\, e^{i\left(\frac{\theta}{2} + \pi\right)}}$$

Q5

$z = \sqrt{3} - \mathrm{i}$, $\operatorname{Im}\left(\frac{z^2}{w}\right) = 0$, $\left|\frac{z^2}{w}\right| = \frac{1}{2}|z|$

Use geometrical reasoning to find the two possibilities for w, giving your answers in exponential form.

[4]

$|z_1 z_2| = |z_1||z_2|$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

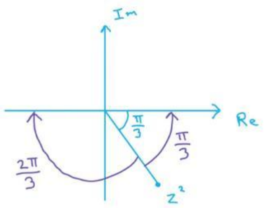
$\left|\frac{z^2}{w}\right| = \frac{|z^2|}{|w|} = \frac{|z|^2}{|w|} = \frac{1}{2}|z| \Rightarrow |w| = 2|z|$

$|z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$ So $|w| = 4$

$\operatorname{Im}\left(\frac{z^2}{w}\right) = 0 \Rightarrow \arg\left(\frac{z^2}{w}\right) = 0$ or π

$z^2 = (\sqrt{3} - i)^2 = 3 - \sqrt{3}i - \sqrt{3}i + i^2 = 2 - 2\sqrt{3}i$

$\arg(z^2) = -\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$



So z^2 has been rotated either $\frac{2\pi}{3}$ radians clockwise or $\frac{\pi}{3}$ radians anticlockwise to get to $\frac{z^2}{w}$.

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

So either $-\arg(w) = -\frac{2\pi}{3} \Rightarrow \arg(w) = \frac{2\pi}{3}$

or $-\arg(w) = \frac{\pi}{3} \Rightarrow \arg(w) = -\frac{\pi}{3}$

$$\boxed{w = 4e^{\frac{2\pi}{3}i} \quad \text{or} \quad 4e^{-\frac{\pi}{3}i}}$$

Q6a

a)

$z = \cos\theta + i\sin\theta$

$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$

$= (-1)\sin\theta + i\cos\theta$

$= i^2 \sin\theta + i\cos\theta$ $\qquad i^2 = -1$

$= i\,(i\sin\theta + \cos\theta)$

$= i\,\underbrace{(\cos\theta + i\sin\theta)}_{=z}$

Therefore

$$\frac{dz}{d\theta} = iz$$

Q6b

b) In general the solution to $\frac{dy}{dx} = ky$ is $y = Ae^{kx}$ (where A is a constant).

Therefore

$$\frac{dz}{d\theta} = iz \Rightarrow z = Ae^{i\theta}$$

$$\Rightarrow e^{i\theta} = \frac{1}{A} z$$

But $z = \cos\theta + i\sin\theta$, so

$$e^{i\theta} = \frac{1}{A}(\cos\theta + i\sin\theta)$$

Let $\theta = 0$, then

$$e^{i(0)} = \frac{1}{A}(\cos(0) + i\sin(0))$$

$$e^{0} = \frac{1}{A}(1 + i(0))$$

$$1 = \frac{1}{A}(1+0) = \frac{1}{A} \Rightarrow A = 1$$

Combining the above,

$$e^{i\theta} = \cos\theta + i\sin\theta$$